Tutorial Laplace 2[1]

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1. Using the Laplace transform solve the differential equation

$$f'' - 4f' + 3f = 1 \tag{1}$$

with boundary conditions $f(0) = f'(0) = 0$.

Solution: First, take the Laplace transform of the equation. Since $f'(0) = f(0) = 0$, if $\mathcal{L}(f) = F(s)$ then $\mathcal{L}(f') = sF(s)$ and $\mathcal{L}(f'') = s^2F(s)$. Thus, the subsidiary equation is

$$s^2F - 4sF + 3F = \frac{1}{s} \tag{2}$$

and so

$$\begin{aligned}(s^2 - 4s + 3)F &= \frac{1}{s} \\ F &= \frac{1}{s}\frac{1}{s^2 - 4s + 3}\end{aligned} \tag{3}$$

and, since $s^2 - 4s + 3 = (s - 3)(s - 1)$, this gives

$$F = \frac{1}{s(s-3)(s-1)} \tag{4}$$

Before we can invert this, we need to do a partial fraction expansion.

$$\begin{aligned}\frac{1}{s(s-3)(s-1)} &= \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-1} \\ 1 &= A(s-3)(s-1) + Bs(s-1) + Cs(s-3)\end{aligned} \tag{5}$$

So substituting in $s = 0$ we get $A = 1/3$, $s = 3$ gives $B = 1/6$ and $s = 1$ gives $C = -1/2$. Hence

$$F = \frac{1}{3s} + \frac{1}{6(s-3)} - \frac{1}{2(s-1)} \tag{6}$$

and so

$$f(t) = \frac{1}{3} + \frac{1}{6}e^{3t} - \frac{1}{2}e^{t} \tag{7}$$

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2. Using the Laplace transform solve the differential equation

$$f'' - 4f' + 3f = 2e^t \tag{8}$$

with boundary conditions $f(0) = f'(0) = 0$.

Solution: This time we have $\mathcal{L}(2e^t) = 2/(s-1)$ on the right hand side. This means that the subsidiary equation is

$$(s^2 - 4s + 3)F = \frac{2}{s-1} \tag{9}$$

so

$$F = \frac{2}{(s-1)^2(s-3)} \tag{10}$$

We need to do partial fractions again, but this is one of those cases with a repeated root:

$$\frac{1}{(s-1)^2(s-3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-3} \tag{11}$$

and multiplying across

$$1 = A(s-1)(s-3) + B(s-3) + C(s-1)^2 \tag{12}$$

so $s = 1$ gives $B = -1/2$ and $s = 3$ gives $C = 1/4$. No value of s gives A on its own, so wee try $s = 2$:

$$1 = -A + \frac{1}{2} + \frac{1}{4} \tag{13}$$

which means that $A = -1/4$. Hence

$$F = -\frac{1}{2(s-1)} - \frac{1}{(s-1)^2} + \frac{1}{2(s-3)} \tag{14}$$

and

$$f = -\frac{1}{2}e^t - te^t + \frac{1}{2}e^{3t} \tag{15}$$

3. Using the Laplace transform solve the differential equation

$$f'' - 4f' + 3f = 0 \tag{16}$$

with boundary conditions $f(0) = 1$ and $f'(0) = 1$.

Solution: In this example there are non-zero boundary conditions. Since

$$\mathcal{L}(f') = sF - f(0) \tag{17}$$

$$\mathcal{L}(f'') = s^2F - sf(0) - f'(0) \tag{18}$$

the subsidiary equation in this case is

$$s^2F - s - 1 - 4sF + 4 + 3F = 0 \tag{19}$$

so

$$(s^2 - 4s + 3)F = s - 3. \tag{20}$$

Hence

$$F = \frac{1}{s-1} \tag{21}$$

and

$$f(t) = e^t \tag{22}$$

4. Using the Laplace transform solve the differential equation

$$y'' - 2ay' + a^2 y = 0 \tag{23}$$

with boundary conditions $y'(0) = 1$ and $y(0) = 0$. a is some real constant.

Solution: Taking the Laplace transform we get

$$s^2 Y - 1 - 2aY + a^2 Y = 0 \tag{24}$$

and hence

$$Y = \frac{1}{(s-a)^2} \tag{25}$$

which means that

$$y = te^{at} \tag{26}$$