Tutorial Laplace 3[1]

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1. Using the Laplace transform solve the differential equation

$$f'' + f' - 6f = e^{-3t} \tag{1}$$

with boundary conditions $f(0) = f'(0) = 0$.

Solution: So, as before, the subsidiary equation is

$$s^2F + sF - 6F = \frac{1}{s+3} \tag{2}$$

or

$$F = \frac{1}{(s+3)^2(s-2)} \tag{3}$$

As before, we do partial fractions

$$\begin{aligned} \frac{1}{(s+3)^2(s-2)} &= \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s-2} \\ 1 &= A(s+3)(s-2) + B(s-2) + C(s+3)^2 \end{aligned} \tag{4}$$

$s = -3$ gives $B = -1/5$ and $s = 2$ gives $C = 1/25$. Putting in $s = 1$ we find

$$1 = -4A + \frac{1}{5} + \frac{16}{25} \tag{5}$$

and so $A = -1/25$. Putting all this together says that

$$f = -\frac{1}{25}e^{-3t} - \frac{t}{5}e^{-3t} + \frac{1}{25}e^{2t} \tag{6}$$

2. Using the Laplace transform solve the differential equation

$$f'' + 6f' + 13f = 0 \tag{7}$$

with boundary conditions $f(0) = 0$ and $f'(0) = 1$. (2)

Solution: So, taking the Laplace transform of the equaiton we get,

$$s^2F - 1 + 6sF + 13F = 0 \tag{8}$$

and, hence,

$$F = \frac{1}{s^2 + 6s + 13}. \tag{9}$$

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Now, using minus b plus or minus the square root of b squared minus four a c all over two a, we get

$$s^2 + 6s + 13 = 0 \tag{10}$$

if

$$s = \frac{-6 \pm \sqrt{36 - 52}}{2} = -3 \pm 2i \tag{11}$$

which means

$$s^2 + 6s + 13 = (s + 3 - 2i)(s + 3 + 2i) \tag{12}$$

Next, we do the partial fraction expansion,

$$\frac{1}{s^2 + 6s + 13} = \frac{A}{s + 3 - 2i} + \frac{B}{s + 3 + 2i} \tag{13}$$

and multiplying across we get

$$1 = A(s + 3 + 2i) + B(s + 3 - 2i) \tag{14}$$

therefore we choose $s = -3 + 2i$ to get

$$A = \frac{1}{4i} = -\frac{i}{4} \tag{15}$$

and $s = -3 - 2i$ to get

$$B = -\frac{1}{4i} = \frac{i}{4} \tag{16}$$

and so

$$F = -\frac{i}{4}\frac{1}{s + 3 - 2i} + \frac{i}{4}\frac{1}{s + 3 + 2i}. \tag{17}$$

If we take the inverse transform

$$\begin{aligned} f &= -\frac{i}{4}e^{-(3-2i)t} + \frac{i}{4}e^{-(3+2i)t} \\ &= \frac{i}{4}e^{-3t}(e^{-2it} - e^{2it}) \\ &= \frac{1}{2}e^{-3t}\sin 2t \end{aligned} \tag{18}$$

3. Using the Laplace transform solve the differential equation

$$f'' + 6f' + 13f = e^t \tag{19}$$

with boundary conditions $f(0) = 0$ and $f'(0) = 0$. (3)

Solution: Taking the Laplace transform of the equation gives

$$s^2F + 6sF + 13F = \frac{1}{s - 1} \tag{20}$$

so that

$$F = \frac{1}{(s-1)(s+3+2i)(s+3-2i)}. \tag{21}$$

We write

$$\frac{1}{(s-1)(s+3+2i)(s+3-2i)} = \frac{A}{s+3-2i} + \frac{B}{s+3+2i} + \frac{C}{s-1} \tag{22}$$

giving

$$1 = A(s-1)(s+3+2i) + B(s-1)(s+3-2i) + C(s+3-2i)(s+3+2i). \tag{23}$$

$s = -3 + 2i$ gives

$$1 = A(-4+2i)(4i) = A(-8-16i) \tag{24}$$

so

$$A = -\frac{1}{8+16i} = -\frac{1}{8+16i}\frac{8-16i}{8-16i} = -\frac{1+2i}{40} \tag{25}$$

In the same way, $s = -3 - 2i$ leads to

$$B = -\frac{1-2i}{40} \tag{26}$$

and, finally, $s = 1$ gives

$$C = \frac{1}{20}. \tag{27}$$

Putting all this together we get

$$F = -\frac{1+2i}{40}\frac{1}{s+3-2i} - \frac{1-2i}{40}\frac{1}{s+3+2i} + \frac{1}{20}\frac{1}{s-1} \tag{28}$$

and so

$$\begin{aligned} f &= -\frac{1+2i}{40}e^{-(3-2i)t} - \frac{1-2i}{40}e^{-(3+2i)t} + \frac{1}{20}e^t \\ &= -\frac{1}{40}e^{-3t}\left[(1+2i)e^{2it} + (1-2i)e^{-2it}\right] + \frac{1}{20}e^t \end{aligned} \tag{29}$$

We then substitute in

$$\begin{aligned} e^{2it} &= \cos 2t + i \sin 2t \\ e^{-2it} &= \cos 2t - i \sin 2t \end{aligned} \tag{30}$$

to end up with

$$f = \frac{1}{20}e^{-3t}[2\sin 2t - \cos 2t] + \frac{1}{20}e^t \tag{31}$$