Tutorial Laplace 2[1]

October 2009

1. (2) Use Laplace transform methods to solve the differential equation

$$f'' + 2f' - 3f = \begin{cases} 1, & 0 \le t < c \\ 0, & t \ge c \end{cases} \tag{1}$$

subject to the initial conditions $f(0) = f'(0) = 0$. (3)

Solution: Taking Laplace transforms of both sides and using the tables for the Laplace transform of the right hand side function, leads to

$$\begin{aligned}
(s^2 + 2s - 3)F &= \frac{1 - e^{-cs}}{s} \\
F &= \frac{1 - e^{-cs}}{s(s^2 + 2s - 3)} \\
&= (1 - e^{-cs}) \frac{1}{s(s-1)(s+3)} \\
&= (1 - e^{-cs}) \left(\frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3} \right) \qquad (2)
\end{aligned}$$

Concentrating on the partial fractions part, we have

$$\begin{aligned}
\frac{1}{s(s-1)(s+3)} &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3} \\
1 &= A(s-1)(s+3) + Bs(s+3) + Cs(s-1) \\
\underline{s = 0:} & \\
1 &= -3A \\
A &= -\frac{1}{3} \\
\underline{s = 1:} & \\
1 &= 0 + 4B + 0 \\
B &= \frac{1}{4} \\
\underline{s = -3:} & \\
1 &= 0 + 012C \\
C &= \frac{1}{12}
\end{aligned}$$

Hence we have

$$F = (1 - e^{-cs}) \left(-\frac{1}{3}\frac{1}{s} + \frac{1}{4}\frac{1}{s-1} + \frac{1}{12}\frac{1}{s+3} \right) \tag{3}$$

[1]UKSW, `tomasik@laic.u-clermont1.fr`, see also `http://laic.u-clermont1.fr/ tomasik/`

From the tables, we know that

$$\mathcal{L}\left(-\frac{1}{3}+\frac{1}{4}e^{t}-\frac{1}{12}e^{-3t}\right)=-\frac{1}{3}\frac{1}{s}+\frac{1}{4}\frac{1}{s-1}+\frac{1}{12}\frac{1}{s+3} \tag{4}$$

and then using the second shift theorem

$$f(t)=-\frac{1}{3}+\frac{1}{4}e^{t}+\frac{1}{12}e^{-3t}-H_c(t)\left(-\frac{1}{3}+\frac{1}{4}e^{(t-c)}+\frac{1}{12}e^{-3(t-c)}\right) \tag{5}$$

2. (3) Use Laplace transform methods to solve the differential equation

$$f''+2f'-3f=\begin{cases}0, & 0\leq t<1\\ 1, & 1\leq t<2\\ 0, & t\geq 2\end{cases} \tag{6}$$

subject to the initial conditions $f(0)=0$ and $f'(0)=0$.

*Solution:*So the thing here is to rewrite the right hand side of the equations in terms of Heaviside functions. Remember the definition of the Heaviside function:

$$H_a(t)=\begin{cases}0 & t<a\\ 1 & t\geq a\end{cases} \tag{7}$$

so the Heaviside function is zero until a and then it is one. The right hand side is zero until $t=1$ and then it is one until $t=2$ and then it is zero again. Consider $H_1(t)-H_2(t)$, this is zero until you reach $t=1$, then the first Heaviside function switches on, the other one remains zero. Things stay like this until you reach $t=2$, then the second Heaviside function switches on aswell and you get $1-1=0$. Thus

$$H_1(t)-H_2(t)=\begin{cases}0, & 0\leq t<1\\ 1, & 1\leq t<2\\ 0, & t\geq 2\end{cases} \tag{8}$$

Now, using

$$\mathcal{L}(H_a(t))=\frac{e^{-as}}{s} \tag{9}$$

we take the Laplace transform of the differential equation:

$$s^2F+2sF-3F=\frac{e^{-s}}{s}-\frac{e^{-2s}}{s} \tag{10}$$

This gives

$$\begin{aligned}(s^2+2s-3)F &= \frac{1}{s}\left(e^{-s}-e^{-2s}\right)\\ F=\frac{1}{s(s-1)(s+3)}\left(e^{-s}-e^{-2s}\right)\end{aligned} \tag{11}$$

Now, if you look at the soln to problem sheet 4, question 3 you'll see that

$$\frac{1}{s(s-1)(s+3)} = -\frac{1}{3s} + \frac{1}{4(s-1)} + \frac{1}{12(s+3)} \tag{12}$$

and we know that

$$\mathcal{L}\left(-\frac{1}{3} + \frac{1}{4}e^{t} + \frac{1}{12}e^{-3t}\right) = -\frac{1}{3} + \frac{1}{4(s-1)} + \frac{1}{12(s+3)} \tag{13}$$

In other word, if it wasn't for the expontentials we'd know the little f. However, we know from the second shift thereom that the affect of the exponential e^{-as} is to change t to $t-a$ and to introduce an overall factor of $H_a(t)$. Thus

$$f = H_1(t)\left(-\frac{1}{3} + \frac{1}{4}e^{t-1} + \frac{1}{12}e^{-3t+3}\right) - H_2(t)\left(-\frac{1}{3} + \frac{1}{4}e^{t-2} + \frac{1}{12}e^{-3t+6}\right) \tag{14}$$

3. (3) Use Laplace transform methods to solve the differential equation

$$f'' + 2f' - 3f = \delta(t-1) \tag{15}$$

subject to the initial conditions $f(0) = 0$ and $f'(0) = 1$.

*Solution:*The only thing that is unusual is that there is a delta function. We take the Laplace transform using

$$\mathcal{L}(\delta(t-a)) = e^{-as} \tag{16}$$

hence

$$(s^2 + 2s - 3)F - 1 = e^{-s} \tag{17}$$

Now, if we do partial fractions on $1/(s^2 + 2s - 3)$ we get

$$\frac{1}{s^2 + 2s - 3} = -\frac{1}{4(s+3)} + \frac{1}{4(s-1)} \tag{18}$$

Hence

$$F = \left(-\frac{1}{4(s+3)} + \frac{1}{4(s-1)}\right)\left(1 + e^{-s}\right) \tag{19}$$

Since

$$\mathcal{L}\left(-\frac{1}{4}e^{-3t} + \frac{1}{4}e^{t}\right) = -\frac{1}{4(s+3)} + \frac{1}{4(s-1)} \tag{20}$$

then, by the second shift theorem we have

$$f = \left(-\frac{1}{4}e^{-3t} + \frac{1}{4}e^{t}\right) + H_1(t)\left(-\frac{1}{4}e^{-3t+3} + \frac{1}{4}e^{t-1}\right) \tag{21}$$