

Tutorial Laplace 5¹

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1. (2) Find the convolution $(f * g)(t)$ when $f(t) = t$, $g(t) = e^{2t}$ ($t \geq 0$).

Solution: From the definition of convolutions

$$\begin{aligned}(f * g)(t) &= \int_0^t f(\tau)g(t - \tau) d\tau = \int_0^t \tau e^{2(t-\tau)} d\tau \\&= \int_0^t \tau e^{2t} e^{-2\tau} d\tau = e^{2t} \int_0^t \tau e^{-2\tau} d\tau \\ \text{Use integration by parts with } &u = \tau, \quad dv = e^{-2\tau} d\tau \\&du = d\tau, \quad v = -\frac{1}{2}e^{-2\tau} \\&= e^{2t} \int_0^t u dv = e^{2t} \left([uv]_0^t - \int_0^t v du \right) \\&= e^{2t} \left(\left[-\frac{\tau}{2} e^{-2\tau} \right]_0^t - \int_0^t -\frac{1}{2} e^{-2\tau} d\tau \right) \\&= e^{2t} \left(-\frac{t}{2} e^{-2t} + 0 + \frac{1}{2} \int_0^t e^{-2\tau} d\tau \right) \\&= -\frac{t}{2} + \frac{e^{2t}}{2} \left[-\frac{1}{2} e^{-2\tau} \right]_0^t \\&= -\frac{t}{2} + \frac{e^{2t}}{2} \left(-\frac{1}{2} e^{-2t} + \frac{1}{2} \right) \\&= -\frac{t}{2} - \frac{1}{4} + \frac{1}{4} e^{2t}\end{aligned}$$

2. (2) Use the convolution theorem to find the function $f(t)$ with

$$\mathcal{L}(f) = \frac{1}{s^2(s-4)}. \quad (1)$$

Solution: We know $\mathcal{L}(t) = \frac{1}{s^2}$ and $\mathcal{L}(e^{4t}) = \frac{1}{s-4}$. From the convolution theorem, we see

$$\mathcal{L}(f) = \frac{1}{s^2(s-4)} = \mathcal{L}(t)\mathcal{L}(e^{4t}) = \mathcal{L}(t * e^{4t})$$

so that $f(t)$ is the convolution $t * e^{4t}$.

$$f(t) = \int_0^t \tau e^{4(t-\tau)} d\tau$$

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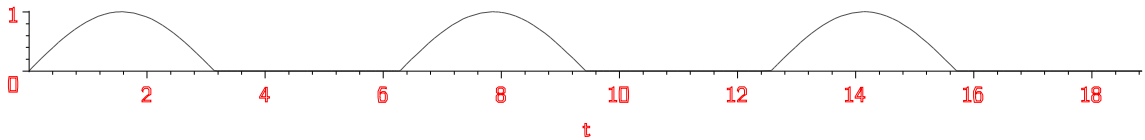
$$\begin{aligned}
&= \int_0^t \tau e^{4t} e^{-4\tau} d\tau = e^{4t} \int_0^t \tau e^{-4\tau} d\tau \\
&\text{Use integration by parts with } U = \tau, \quad dV = e^{-4\tau} d\tau \\
&\quad dU = d\tau, \quad V = -\frac{1}{4} e^{-4\tau} \\
&= e^{4t} \int_0^t U dV = e^{4t} \left([UV]_0^t - \int_0^t V dU \right) \\
&= e^{4t} \left(\left[-\frac{\tau}{4} e^{-4\tau} \right]_0^t - \int_0^t -\frac{1}{4} e^{-4\tau} d\tau \right) \\
&= e^{4t} \left(-\frac{t}{4} e^{-4t} + 0 + \frac{1}{4} \int_0^t e^{-4\tau} d\tau \right) \\
&= -\frac{t}{4} + \frac{e^{4t}}{4} \left[-\frac{1}{2} e^{-4\tau} \right]_0^t \\
&= -\frac{t}{4} + \frac{e^{4t}}{4} \left(-\frac{1}{4} e^{-4t} + \frac{1}{4} \right) \\
&= -\frac{t}{4} - \frac{1}{16} + \frac{1}{16} e^{4t}
\end{aligned}$$

3. (4) Use the formula for the Laplace transform of a periodic function with period c :

$$\mathcal{L}(f) = \frac{1}{1 - e^{-cs}} \int_0^c f(t) e^{-st} dt \quad (2)$$

to find the Laplace transform of a half-rectified wave

$$f(t) = \begin{cases} \sin t & \sin t > 0 \\ 0 & \sin t \leq 0 \end{cases} \quad (3)$$



This is the form a AC current has after going through a diode.

We substitute this into the formula

$$\mathcal{L}(f) = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} f(t) e^{-st} dt = \frac{1}{1 - e^{-2\pi s}} \int_0^{\pi} \sin t e^{-st} dt \quad (4)$$

We need to do the integral. There are two obvious ways, the first is to split the sine into exponentials

$$\begin{aligned}
\int_0^{\pi} \sin t e^{-st} dt &= \frac{1}{2i} \left(\int_0^{\pi} e^{(i-s)t} dt - \int_0^{\pi} e^{-(i+s)t} dt \right) \\
&= \frac{1}{2i} \left[\frac{1}{i-s} (e^{(i-s)\pi} - 1) + \frac{1}{i+s} (e^{-(i+s)\pi} - 1) \right] \quad (5)
\end{aligned}$$

Now, we use

$$e^{i\pi} = e^{-i\pi} = -1 \quad (6)$$

and

$$\begin{aligned} \frac{1}{i-s} &= \frac{1}{i-s} \frac{-i-s}{-i-s} = -\frac{s+i}{s^2+1} \\ \frac{1}{i+s} &= \frac{1}{i+s} \frac{-i+s}{-i+s} = \frac{s-i}{s^2+1} \end{aligned} \quad (7)$$

to get

$$\int_0^\pi \sin te^{-st} dt = \frac{1+e^{-s\pi}}{1+s^2} \quad (8)$$

or

$$\mathcal{L}(f) = \frac{1}{s^2+1} \frac{1+e^{-s\pi}}{1-e^{-2s\pi}} = \frac{1}{s^2+1} \frac{1}{1-e^{-s\pi}} \quad (9)$$

where the final equality uses

$$1 - e^{-2s\pi} = (1 - e^{-s\pi})(1 + e^{-s\pi}) \quad (10)$$

The other way to do the integral is to integrate by parts. Briefly, write

$$\begin{aligned} I = \int_0^\pi \sin te^{-st} dt &= -\frac{1}{s} \int_0^\pi \cos te^{-st} dt \\ &= -\frac{1}{s} \left[-\frac{1}{s} (e^{-\pi s} + 1) + \frac{1}{s} I \right] \end{aligned} \quad (11)$$

and solve for I to get the answer given at (8) above.