Finite structures of arithmetics and reduced products

Michał Krynicki
Cardinal Stefan Wyszyński University in Warsaw
Jerzy Tomasik
IUT, Universite d'Auvergne, Clermont–Ferrand
Konrad Zdanowski
Institute of Mathematics, Polish Academy of Science, Warsaw

In [M1] Marcin Mostowski proposed to consider a theory of initial segments of standard models of arithmetic. In this approach one replaces an actual infinity of a standard model with the family of finite models, which can be seen as potentialy infinite. In [M1] and [M2], he introduces a fundamental concepts and describes the semantical strenth of such theory. Further results on this subject are also done in [KZ] and [MW].

In our paper we consider a relation between theories of finite structures of arithmetic and the ultraproduct construction.

1 Basic definitions

In this section we fix the notation and introduce the main concepts.

Let $\mathcal{A}$ be a model having as a universe the set of natural numbers, i.e. $\mathcal{A} = (\mathbf{N}, R_1, \ldots, R_s, f_1, \ldots, f_t, a_1, \ldots, a_r)$, where $R_1, \ldots, R_s$ are relations on $\mathbf{N}$, $f_1, \ldots, f_t$ are operations (not necessarily unary) on $\mathbf{N}$ and $a_1, \ldots, a_r \in \mathbf{N}$. We will consider finite initial fragments of these models. Namely, for $n \in \mathbf{N}$, by $\mathcal{A}_n$ we denote the following structure

$$\mathcal{A}_n = (\{0, \ldots, n\}, R_1^n, \ldots, R_s^n, f_1^n, \ldots, f_t^n, a_1^n, \ldots, a_r^n, n),$$

where R_i^n is the restriction of R_i to the set $\{0, \ldots, n\}$, f_i^n is defined as

$$f_i^n(b_1, \ldots, b_{n_i}) = \begin{cases} f_i(b_1, \ldots, b_{n_i}) & \text{if } f(b_1, \ldots, b_{n_i}) \leq n \\ n & \text{if } f(b_1, \ldots, b_{n_i}) > n \end{cases}$$

and $a_i^n = a_i$ if $a_i \leq n$, otherwise $a_i^n = n$. We will denote the family $\{\mathcal{A}_n\}_{n \in \mathbf{N}}$ by $FM(\mathcal{A})$.

The signature of $\mathcal{A}_n$ is an extension of the signature of $\mathcal{A}$ by one constant. This constant will be denoted by MAX.

Let $\varphi(x_1, \ldots, x_p)$ be a formula and $b_1, \ldots, b_p \in \mathbf{N}$. We say that φ *is satisfied by* $b_1, \ldots, b_p$ *in all finite models of* $FM(\mathcal{A})$ ($FM(\mathcal{A}) \models \varphi[b_1, \ldots, b_p]$) if for all $n \geq max(b_1, \ldots, b_p)$ $\mathcal{A}_n \models \varphi[b_1, \ldots, b_p]$.

We say that φ *is satisfied by* $b_1, \ldots, b_p$ *in all sufficiently large finite models of* $FM(\mathcal{A})$, what is denoted by $FM(\mathcal{A}) \models_{sl} \varphi[b_1, \ldots, b_p]$, if there is $k \in \mathbf{N}$ such that for all $n \geq k$ $\mathcal{A}_n \models \varphi[b_1, \ldots, b_p]$.

When no ambiguity arises we will use $\models_{sl} \varphi[b_1, \ldots, b_p]$ instead of $FM(\mathcal{A}) \models_{sl} \varphi[b_1, \ldots, b_p]$.

Finally, a sentence φ *is true in all finite models of* $FM(\mathcal{A})$ if $\mathcal{A}_n \models \varphi$ for all $n \in \mathbf{N}$. Similarly, a sentence φ *is true in all sufficiently large finite models of* $FM(\mathcal{A})$ if there is $k \in \mathbf{N}$ such that for all $n \geq k$ $\mathcal{A}_n \models \varphi$.

By $Th(\mathcal{A})$, where $\mathcal{A}$ is a structure, we denote the set of all sentences true in $\mathcal{A}$. For a class of models $\mathcal{K}$, by $Th(\mathcal{K})$ we denote the set of sentences true in all models from $\mathcal{K}$, that is $Th(\mathcal{K}) = \bigcap_{\mathcal{A} \in \mathcal{K}} Th(\mathcal{A})$.

By $sl(\mathcal{A})$ we denote the set of sentences true in all sufficiently large finite models of $FM(\mathcal{A})$ i.e.

$$sl(\mathcal{A}) = \{\varphi \ : \exists k \forall n \geq k \ \mathcal{A}_n \models \varphi\}.$$

Sometimes we will use the set

$$sl^-(\mathcal{A}) = \{\varphi \in sl(\mathcal{A}) \ : \ \varphi \text{ is of a signature of } \mathcal{A}\}$$

Our aim is to investigate the properties of $sl(\mathcal{A})$ for different models $\mathcal{A}$ and compare them with properties of the reduced products of structures $\mathcal{A}_n$.

The upper and lower limits of a sequence of sets $\{X_n\}_{n \in \omega}$ are defined as follows (see e.g. [KM]):

$\liminf_{n \to \infty} X_n = \bigcup_{n \in \omega} \bigcap_{k \in \omega} X_{n+k}$

$\limsup_{n \to \infty} X_n = \bigcap_{n \in \omega} \bigcup_{k \in \omega} X_{n+k}$

Obviously, $\liminf_{n \to \infty} X_n \subseteq \limsup_{n \to \infty} X_n$. If, instead the inclusion, the equality holds then we say that the sequence of sets $\{X_n\}_{n \in \omega}$ converges and its limit is equal to the upper or lower limit.

We can characterize the set $sl(\mathcal{A})$ in terms of a limit of a sequence of sets of sentences.

Fact 1.1 $sl(\mathcal{A}) = \liminf_{n \to \infty} Th(\mathcal{A}_n)$.

In some places of our paper we will need also the notion of FM-representability introduced in [M1]. A relation $R \subseteq \mathbf{N}^p$ is FM–representable in $FM(\mathcal{A})$ if and only if there exists a formula $\varphi(x_1, \ldots, x_p)$ such that for all $a_1, \ldots, a_p \in \mathbf{N}$,

$$(a_1, \ldots, a_p) \in R \text{ if and only if } FM(\mathcal{A}) \models_{sl} \varphi[a_1, \ldots, a_p]$$

and

$$(a_1, \ldots, a_p) \notin R \text{ if and only if } FM(\mathcal{A}) \models_{sl} \neg\varphi[a_1, \ldots, a_p].$$

2 Basic properties of $sl(\mathcal{A})$

At the first let us observe that the set of sentences $sl(\mathcal{A})$ is a closed under logical inferences.

Fact 2.1 *If $\varphi \in sl(\mathcal{A})$ and $\varphi \Rightarrow \psi \in sl(\mathcal{A})$ then $\psi \in sl(\mathcal{A})$. In particular, if $\varphi \in sl(\mathcal{A})$ and $\models \varphi \Rightarrow \psi$ then $\psi \in sl(\mathcal{A})$.*

From the above fact it follows that the theory $sl(\mathcal{A})$ is consistent and, hence, has a model. Indeed, assume that for some $\varphi \in Cn(sl(\mathcal{A}))$ also $\neg\varphi \in Cn(sl(\mathcal{A}))$ then from the fact 2.1 $\varphi \in sl(\mathcal{A})$ and $\neg\varphi \in sl(\mathcal{A})$ what is impossible.

Observe that every sentence of the form $\exists^{\geq n} x(x = x)$ belongs to $sl(\mathcal{A})$. Thus every model for $sl(\mathcal{A})$ is infinite.

Examples

1. Let $\mathcal{A}$ be a structure of the empty signature. So, the theory $sl(\mathcal{A})$ is simply the theory of infinite structures. Hence it is a complete theory. Moreover, $sl^-(\mathcal{A}) = Th(\mathcal{A})$.

2. Let $\mathcal{A} = (\mathbf{N}, \leq)$ or $\mathcal{A} = (\mathbf{N}, <)$. In these cases $sl(\mathcal{A})$ is the theory of a linear discrete ordering with the first and last elements, i.e. $sl(\mathcal{A}) = Th((\omega + \omega^*, \leq, m)$ or $sl(\mathcal{A}) = Th((\omega + \omega^*, <, m)$, respectively, where m is the last element. The theory $sl^-(\mathcal{A})$ is also a complete theory. However, $sl(\mathcal{A}) \neq Th(\mathcal{A})$ because $\mathcal{A} \models \forall x \exists y(x < y)$ and obviously $\forall x \exists y(x < y) \notin sl(\mathcal{A})$.

3. Let $\mathcal{A} = (\mathbf{N}, S)$, where S is a successor function. As in the previous example the theory $sl(\mathcal{A})$ is a complete theory. Using the Ehrenfeucht–Fraisse games, it can be proved that $sl(\mathcal{A}) = Th((\omega + \omega^*, S^*, m))$, where m is the last element and $S^*(x) = y$ holds if y is an immediate successor of x or $x = y = m$. In this case also the theories $sl^-(\mathcal{A})$ and $Th(\mathcal{A})$ differ from each other. For example a very simple sentence $\forall x(S(x) \neq x)$ belongs to $Th(\mathcal{A})$ and does not belong to $sl(\mathcal{A})$.

4. Let $\mathcal{A} = (\mathbf{N}, +)$. Let φ be the following sentence: $\exists x(x + x \neq MAX \wedge x + x + 1 = MAX)$. In this case neither φ nor $\neg\varphi$ belong to the $sl(\mathcal{A})$. So, $sl((\mathbf{N}, +))$ is not complete. Later we will give several examples of complete extensions of $sl((\mathbf{N}, +))$.

5. Let $\mathcal{A} = (\mathbf{N}, \times)$. In that case $sl(\mathcal{A})$ is not complete. In [KZ] there is an example of a sentence φ which is true in $\mathcal{A}_n$ if and only if n is a square of a natural number. So, $\varphi \notin sl(\mathcal{A})$ and $\neg\varphi \notin sl(\mathcal{A})$.

6. Let $\mathcal{N} = (\mathbf{N}, +, \times)$. Here, again the theory $sl(\mathcal{N})$ is not complete. To see this it is enough to take the formula φ from one of the last three examples. For more informations on complete extensions of $sl(\mathcal{N})$ see sections 6 and 7.

The above examples show that, in general, the relation between $Th(\mathcal{A})$ and $sl(\mathcal{A})$ is rather weak. Even Π_1–sentences from $Th(\mathcal{A})$ may not belong to $sl(\mathcal{A})$ (see example 3). We can state only the following.

Fact 2.2 *If φ is a Σ_1–sentence and $\varphi \in Th(\mathcal{A})$ then $\varphi \in sl(\mathcal{A})$.*

As the above examples show, in general, $sl(\mathcal{A})$ is not a complete theory. This corresponds with our next observation stating that there is no relation between the relation of elemetary equivalence of models for arithmetic and the equality relation between theories of suficiently large finite models.

Fact 2.3 *There are structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \not\equiv \mathcal{B}$ and $sl(\mathcal{A}) = sl(\mathcal{B})$.*

Proof. Let R and S be a linear ordering on $\mathbf{N}$ of the type $\omega + \omega^*$ and $\omega + \omega + \omega^*$ respectively. Obviously, structures $(\mathbf{N}, R)$ and $(\mathbf{N}, S)$ are not elementary equivalent. But as we can observe $sl((\mathbf{N}, R)) = sl((\mathbf{N}, S))$. Really, let $T = Th((\omega + \omega^*, <)) \cup \{\exists^{\geq n} x(x > MAX) \ : \ n \in \mathbf{N}\} \cup \{\exists^{\geq n} x(x < MAX) \ : \ n \in \mathbf{N}\}$. Using Ehrenfeucht–Fraisse games we can prove that every two models for T are elementary equivalent. So, T is a complete theory. From the other side it is easy to verify that $T \subseteq sl((\mathbf{N}, R)) \cap sl((\mathbf{N}, S))$. **Q.E.D**

Fact 2.4 *There are structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \cong \mathcal{B}$ and $sl(\mathcal{A}) \neq sl(\mathcal{B})$.*

Proof. Let R be the divisibility relation on $\mathbf{N}$. Let $f \in \mathbf{N}^{\mathbf{N}}$ be defined as follows:

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is prime} \\ \frac{x}{2} & \text{if } x = 2p, \text{ where } p \text{ is prime} \\ x & \text{in other cases} \end{cases}$$

Let S be the image of R in f, i.e. $S = \{(f(x), f(y)) : (x, y) \in R\}$. We put $\mathcal{A} = (\mathbf{N}, R, 0, 1)$ and $\mathcal{B} = (\mathbf{N}, S, 0, 1)$. Obviously, $\mathcal{A} \cong \mathcal{B}$. But the sentence $\exists x \forall y[(P(x, y) \leftrightarrow (x = y \vee y = 0)) \wedge (P(y, x) \leftrightarrow (x = y \vee y = 1))]$ belongs to $sl(\mathcal{A})$ and does not belongs to $sl(\mathcal{B})$. So, $sl(\mathcal{A}) \neq sl(\mathcal{B})$. **Q.E.D**

In the above proof we gave an example of the structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \cong \mathcal{B}$ and $sl(\mathcal{A}) \neq sl(\mathcal{B})$. It could be strange that the operation sl does not preserve isomorphism. Let us note that in the definition of this operation important role play the standard ordering on $\mathbf{N}$. This ordering is not definable in the structures considered in the proof of the fact 2.4. This is why such case is possible.

Proposition 2.5 *For every structure $\mathcal{A}$, $sl(\mathcal{A})$ is not finitely axiomatizable.*

Proof. It is known (see [FMS]) that if a theory T is finitely axiomatizable then the class of structures $Mod(T)$ as well as their complement is closed under ultraproduct construction. As we observed, every model of $sl(\mathcal{A})$ is infinite. So, for each n, $\mathcal{A}_n \notin Mod(sl(\mathcal{A}))$. On the other side, by Łoś theorem (see [Ł]), for arbitrary nonprincipal ultrafilter U in $\mathbf{N}$ $\prod_{n\in\omega} \mathcal{A}_n / U$ is a model for $sl(\mathcal{A})$. **Q.E.D**

3 Reduced product of finite structures

Let Fr denote the Frechet filter, i.e. $Fr = \{X \subseteq \mathbf{N} : \mathbf{N} - X \text{ is finite}\}$. It is natural to expect that there exists some relation between $sl(\mathcal{A})$ and the theory of reduced product of structures from the family $FM(\mathcal{A})$. As the following examples show such relations are rather very weak.

Examples

1. If $\mathcal{A}$ is a structure of the empty signature then obviously $Th(\prod_{n\in\omega} \mathcal{A}_n / Fr) = sl(\mathcal{A})$.

2. Let $\mathcal{A} = (\mathbf{N}, \leq)$. As we observed $sl(\mathcal{A})$ is the theory of the linear discreate ordering with the first and last elements. On the other hand $\prod_{n\in\omega}(\mathcal{A}_n / Fr)$ is not a linear ordering. It is a dense lattice with the least and greatest elements.

3. Let $\mathcal{A} = (\mathbf{N}, <)$. In this case $\prod_{n\in\omega}(\mathcal{A}_n / Fr)$ is also a nonlinear ordering having continuum many minimal and maximal elements. So, in this case, the theories $sl(\mathcal{A})$ and $Th(\prod_{n\in\omega}(\mathcal{A}_n / Fr))$ are quite different.

4. Let $\mathcal{A} = (\mathbf{N}, +)$. In this case the theory $Th(\prod_{n\in\omega}(\mathcal{A}_n/Fr))$ is also different than $sl(\mathcal{A})$ because the formula which defines a linear ordering in the last theory does not define such ordering in the first case.

5. Let $\mathcal{A} = (\mathbf{N}, \times)$. As we can easily see, there are zero's divisors in $\prod_{n\in\omega}(\mathcal{A}_n/Fr)$. Obviously, the sentence $\forall x\forall y(x \times y = 0 \rightarrow (x = 0 \vee y = 0))$ belongs to $sl(\mathcal{A})$ (observe that zero is definable in both theories). So, again the theory of the reduced product of finite structures is not the same as $sl(\mathcal{A})$.

5. Let $\mathcal{A} = (\mathbf{N}, |)$. In that case $sl(\mathcal{A})$ contains axioms of the theory of partial ordering which is not dense but $\prod_{n\in\omega}(\mathcal{A}_n/Fr)$ is a dense partial ordering.

The above examples show that the theory of reduced product of finite structures usually is quite different than the theory of sentences true in sufficiently large finite structures.

However, using well know results concerning the reduced product construction, we can observe the following.

Proposition 3.1 *If φ is a Horn sentence and $\varphi \in sl(\mathcal{A})$ then $\varphi \in Th(\prod_{n\in\omega}(\mathcal{A}_n/Fr))$.*

4 Complete extensions of $sl(\mathcal{A})$

As we mentioned in the proof of the Proposition 2.5 for arbitrary nonprincipal ultrafilter U in $\mathbf{N}$, $\prod_{n\in\omega}\mathcal{A}_n/U$ is a model for $sl(\mathcal{A})$. It means that for arbitrary nonprincipal ultrafilter U on $\mathbf{N}$ the theory $Th(\prod_{n\in\omega}\mathcal{A}_n/U)$ is a complete extension of the theory $sl(\mathcal{A})$. We state something more. We start with an easy but usefull observation.

Fact 4.1 *For each sentence φ, φ is consistent with $sl(\mathcal{A})$ if and only if for arbitrary large n, $\mathcal{A}_n \models \varphi$.*

Corollary 4.2 *a) For each sentence φ, φ is consistent with $sl(\mathcal{A})$ if and only if $\varphi \in \limsup_{n\to\infty} Th(\mathcal{A}_n)$*

b) $sl(\mathcal{A})$ is complete if and only if the sequence $(Th(\mathcal{A}_n))_{n\in\mathbf{N}}$ converge.

Definition 4.3 *Let $X \subseteq \mathbf{N}$. By $FM_X(A)$ we define the family*

$$\{\mathcal{A}_n \in FM(\mathcal{A}) \ : \ n \in X\}.$$

Similarly, we can generalize the notion of sl–theory to a family $FM_X(\mathcal{A})$.

$$sl_X(\mathcal{A}) = \{\varphi \ : \ \exists k\forall n \geq k(\mathcal{A}_n \in FM_X(\mathcal{A}) \Rightarrow \mathcal{A}_n \models \varphi)\}.$$

When $X = \mathbf{N}$, then $sl_X(\mathcal{A})$ is just $sl(\mathcal{A})$ but in general $sl_X(\mathcal{A})$ could contain more sentences. Obviously for arbitrary set of natural numbers X the set of sentences $sl_X(\mathcal{A})$ has similar logical properties as $sl(\mathcal{A})$. In particular, the properties expressed in the fact 2.1. As we will show, any complete extension T of $sl(\mathcal{A})$ can be characterized by a suitable X.

Immediately from the definition and Łoś theorem follows also the following fact.

Fact 4.4 *For arbitrary infinite set $X \subseteq \mathbf{N}$ and nonprincipal ultrafilter U on $\mathbf{N}$ such that $X \in U$, $sl_X(\mathcal{A}) \subseteq Th(\prod_{n\in\mathbf{N}} \mathcal{A}_n/U)$.*

Theorem 4.5 *For $\mathcal{A}$ and any complete extension T of $sl(\mathcal{A})$ one can choose an X such that $T = sl_X(\mathcal{A})$. Moreover, X is recursive in T.*

Proof. Let $T = \{\varphi_i\}_{i\in\mathbf{N}}$ be a complete extension of $sl(\mathcal{A})$. We will construct a sequence of integers $\{x_i\}_{i\in\mathbf{N}}$ such that

- $x_i < x_{i+1}$, for $i \in \mathbf{N}$,
- for all $i \in \mathbf{N}$, for all $j \geq i$, $\mathcal{A}_{x_j} \models \bigwedge_{k\leq i} \varphi_i$.

It follows that for $X = \{x_i\}_{i\in\mathbf{N}}$, $sl_X(\mathcal{A}) = T$.

We can construct $\{x_i\}_{i\in\mathbf{N}}$ as follows:

- $x_0 = 0$
- $x_{i+1} = \min\{n \ : \ x_i < n \wedge \mathcal{A}_n \models \bigwedge_{j\leq i+1} \varphi_i\}$.

Since T is a consistent extension of $sl(\mathcal{A})$, for each i, the sentence $\bigwedge_{k\leq i} \varphi_k$ has arbitrary large finite models in $FM(\mathcal{A})$ (see fact 4.1). Therefore, $\{x_i\}_{i\in\mathbf{N}}$ is a well defined infinite sequence of integers satisfying all needed properties. **Q.E.D**

Let us observe that if T is a complete extension of $sl(\mathcal{A})$ and X is a set constructed in the proof of theorem 4.5 then for any nonprincipal ultrafilter U such that $X \in U$, $T = Th(\prod_{n\in\mathbf{N}} \mathcal{A}_n/U)$. So, we have the following.

Proposition 4.6 *For arbitrary complete extension T of $sl(\mathcal{A})$ there exists a nonprincipal ultrafilter U such that $T = Th(\prod_{n\in\mathbf{N}} \mathcal{A}_n/U)$.*

From the proposition 4.6 and the theorem 4.5 we have

Corollary 4.7 $sl(\mathcal{A}) = \bigcap_{U\supset Fr} Th(\prod_{n\in\mathbf{N}} \mathcal{A}_n/U)$.

5 Complete extensions of $sl((\mathbf{N},+))$

Now, we will consider an example of $\mathcal{A} = (\mathbf{N}, +)$. So, in whole this section, $\mathcal{A}$ denotes the structure $(\mathbf{N}, +)$.

Let, for $n > 0$, $\Theta(n)$ denote the following sentence

$$\exists x(\underbrace{(x+1)+\ldots+(x+1)}_{n} = MAX \wedge \underbrace{(x+1)+\ldots+(x+1)}_{n-1} + x \neq MAX)$$

Obviously for every $n > 1$, $\Theta(n) \notin sl(\mathcal{A})$ and $\neg\Theta(n) \notin sl(\mathcal{A})$.

Let $(a_n)_{n\in\mathbf{N}}$ be a sequence such that for each $n \in \mathbf{N}$ $a_n \in \mathbf{N} \cup \{\omega\}$. For such a sequence $(a_n)_{n\in\mathbf{N}}$ we define the following set of sentences

$$T_{(a_n)} = \{\Theta(p_n^{a_n}) \wedge \neg\Theta(p_n^{a_n+1}) \,:\, a_n \neq \omega\} \cup \{\Theta(p_n^k) \,:\, k \in \mathbf{N} \text{ and } a_n = \omega\}$$

Fact 5.1 *If a sequence $(a_n)_{n\in\mathbf{N}}$ is such that for some $i \in \mathbf{N}$ $a_i = \omega$ or for infinitely many i $a_i \neq 0$ then $sl(\mathcal{A}) \cup T_{(a_n)}$ is consistent set of sentences.*

Proof. Let $(n_j^i)_{i,j\in\mathbf{N},j<i}$ be an indexed set of natural numbers defined as follows: if $a_j \neq \omega$ then for each $i > j$ $n_j^i = a_j$ and if $a_j = \omega$ then for each $i > j$ $n_j^i = i$. We put $X = \{\prod_{j<i} p_j^{n_j^i}\}_{i\in\mathbf{N}}$. An easy verification shows that $T_{(a_n)} \subseteq sl_X(\mathcal{A})$ what shows that $T_{(a_n)}$ is a consistent set of sentences. **Q.E.D**

There is a continuum many sequences satisfying assumption of the fact 5.1. obviously for two different sequences $(a_n)_{n\in\mathbf{N}}$, $(b_n)_{n\in\mathbf{N}}$ the set of sentences $T_{(a_n)} \cup T_{(b_n)}$ is inconsistent. Thus we have

Corollary 5.2 *$sl(\mathbf{A})$ has a continuum complete extensions.*

6 Complete extensions of $sl(\mathcal{N})$

Now, we will consider the standard model of arithemetic $\mathcal{N} = (\mathbf{N}, +, \cdot, 0, 1)$. We will show that there is a formula $\varphi(x)$ such that it can FM–represent any subset of $\mathbf{N}$, in some complete extension of $sl(\mathcal{N})$. We can think about φ as a formula which is undetermined for any $a \in \mathbf{N}$. Firstly, we need the following definition.

Definition 6.1 *The 2–ary pairing function, $\langle .,. \rangle_2 : \mathbf{N}^2 \longrightarrow \mathbf{N}$, is defined as $\langle x, y\rangle_2 = \frac{(x+y)(x+y+1)}{2} + y$. It is a bijection between $\mathbf{N}^2$ and $\mathbf{N}$.*

By induction, we define a d–ary pairing function for $d \geq 2$. If $\langle \ldots \rangle_d : \mathbf{N}^d \longrightarrow \mathbf{N}$ is defined then $\langle \ldots \rangle_{d+1} : \mathbf{N}^{d+1} \longrightarrow \mathbf{N}$ is defined as

$$\langle x_1, \ldots, x_{d+1}\rangle_{d+1} = \langle x_1, \langle x_1, \ldots, x_d\rangle_d\rangle_2.$$

Usually the index d will be omitted.

Let us observe, that for each d the graph of the pairing function $\langle \ldots \rangle_d$ is Δ_0 definable. It follows that we can define the proper restriction of its graph in each finite model from $FM(\mathcal{N})$.

Theorem 6.2 *There exists $\varphi(x)$ such that for each $A \subseteq \mathbf{N}$ there is an ulrafilter $\mathcal{U}$ such that*

$$A = \{a \in \mathbf{N} : \Pi_{n \in \mathbf{N}} \mathcal{N}_n / \mathcal{U} \models \varphi[\mathbf{a}]\},$$

where $\mathbf{a}$ *is defined as an equivalence class of the function*

$$f_a(i) = \begin{cases} 0 & \textit{if } i < a, \\ a & \textit{otherwise.} \end{cases}$$

Proof. To start with, we will define the family of sets $\{X_i\}_{i \in \mathbf{N}}$ such that

- $X_i \subseteq \mathbf{N}$,
- for each sequence $n_1, \ldots, n_{k+m}$ of pairwise different integers

  $$\bigcap_{1 \leq i \leq k} X_{n_i} \cap \bigcap_{1 \leq i \leq m} (\mathbf{N} - X_{n_{k+i}})$$

  is infinite.

We take

$$X_i = \{x : \exists z_1 \leq x \exists z_2 \leq x (x = \langle z_1 p_{i+1}^{p_{i+1}}, z_2 \rangle)\},$$

where p_k is the k-th prime number. To see that $\{X_i\}_{i \in \omega}$ has the desired properties let $n_1, \ldots, n_{k+m}$ be a sequence of pairwise different integeres. Then $\bigcap_{1 \leq i \leq k} X_{n_i} \cap \bigcap_{1 \leq i \leq m} (\mathbf{N} - X_{n_{k+i}})$ contains each number $\langle \Pi_{i \leq k} p_{n_i}^{p_{n_i}}, y \rangle$. This property of the family $\{X_i\}_{i \in \mathbf{N}}$ guarantees that for each $\varepsilon : \mathbf{N} \longrightarrow \{0, 1\}$ the family $\{X_i^{\varepsilon(i)}\}_{i \in \mathbf{N}}$, where

$$X_i^a = \begin{cases} X_i & \text{if } a = 1, \\ \mathbf{N} - X_i & \text{if } a = 0, \end{cases}$$

will have the finite intersection property. Consequently, there will be an ultrafilter in which this family is contained.

By $pr(x, y)$ we will denote the functional relation $y = p_x^{p_x}$. pr is Δ_0 definable so its graph is uniformly definable in each finite model $\mathbf{N}_i$.

It is not known whether the relation "y is the x-th prime" is Δ_0. However, we can Δ_0 "compute" p_x in $pr(x, y)$ using y as a bound for quantifiers in $pr(x, y)$.

Let $\varphi'(x, y)$ be a formula which defines in each model from $FM(\mathcal{N})$ the following relation

$$\exists z \exists z_1 \exists z_2 (pr(x+1, z) \wedge y = \langle z_1 z, z_2 \rangle).$$

Then, as $\varphi(x)$ we take $\varphi'(x, MAX)$. Let us observe that, for $i \in \mathbf{N}$, $X_i = \{k \in \mathbf{N} : \mathbf{N}_k \models \varphi[i]\}$.

Now we will show that φ satisfies the assertion of the theorem. Let $A \subseteq \mathbf{N}$ and let $\xi_A : \mathbf{N} \longrightarrow \{0, 1\}$ be the characteristic fuction of A,

$$\xi_A(i) = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a nonprincipial ultrafilter $\mathcal{U}$ containing the family $\{X_i^{\xi_A(i)}\}_{i \in \mathbf{N}}$ constructed from this ξ_A. For proving the theorem one should observe that the following are equivalent for each $a \in \mathbf{N}$,

$$\begin{aligned} a \in A &\iff X_a^{\xi_A(a)} \in \mathcal{U} \\ &\iff \{k \ : \ \mathcal{N}_k \models \varphi[a]\} \in \mathcal{U} \\ &\iff \Pi_{i \in \mathbf{N}} \mathcal{N}_{i/\mathcal{U}} \models \varphi[[f_a]]. \end{aligned}$$

Q.E.D

The theorem 6.2 allows us also to show that there is as many complete extensions of $sl(\mathcal{N})$ as it is possible.

Theorem 6.3 *There is continuum complete, consistent extensions of $sl(\mathbf{N})$.*

Proof. Let φ be as in theorem 6.2. Then for each $X \subseteq \mathbf{N}$, there is an ultrafilter $\mathcal{U}_X$ such that φ defines X in $\Pi_{n \in \mathbf{N}} \mathcal{N}_n / \mathcal{U}_X$. Of course, for different subsets of $\mathbf{N}$, X and Y, theories of models $\mathcal{A}_X = \Pi_{n \in \mathbf{N}} \mathcal{N}_n / \mathcal{U}_X$ and $\mathcal{A}_Y = \Pi_{n \in \mathbf{N}} \mathcal{N}_n / \mathcal{U}_Y$ are different because φ defines X in $\mathcal{A}_X$ and Y in $\mathcal{A}_Y$. Since there is continuum different subsets of $\mathbf{N}$ the theorem is proven. **Q.E.D**

Observe that the above theorem follows also from the corollary 5.2

7 Complexity of complete extensions of $sl(\mathcal{N})$

In fourth sections we have shown that for any $\mathcal{A}$ and for any recursive, and complete extension T of $sl(\mathcal{A})$ one can find a recursive X such that $T = sl_X(\mathcal{A})$. In this section we show that if we consider a model $\mathbf{N} = (\mathbf{N}, +, \times)$ then the complexity of X such that $sl_X(\mathcal{N})$ is a complete theory is not in

$\Pi_1 \cup \Sigma_1$ in the arithmetical hierarchy. Similarly, any such extension is of complexity at least Δ_3.

By a standard way we can prove the following fact.

Fact 7.1 *For any complete theory T and for any n, if $T \in \Sigma_n$, then $T \in \Delta_n$.*

Fact 7.2 *For any $X \subseteq \mathbf{N}$, $sl_X(\mathcal{N}) \notin \Delta_2$.*

Proof. Assume that $sl_X(\mathcal{N}) \in \Delta_2$. All sets which are in Δ_2 are FM–representable in $FM(\mathcal{A})$. It follows that they are representable in any complete extension of $sl(\mathcal{N})$. However, if such extension is in Δ_2, then it would define the truth for itself what is impossible by Tarski's theorem. **Q.E.D**

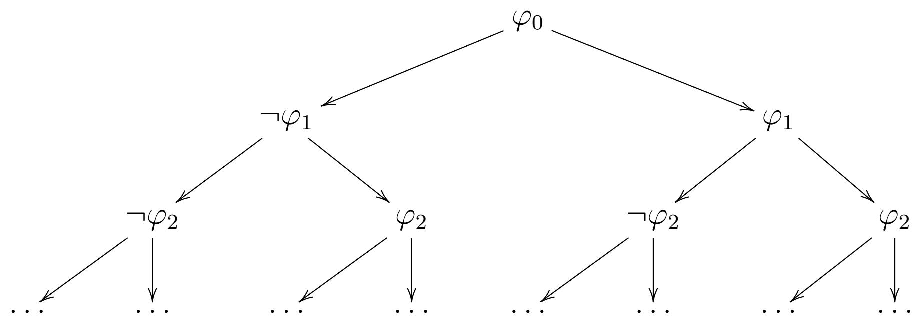
By the above facts we have the following conclusion.

Corollary 7.3 *The complexity of a complete extension of $sl(\mathcal{N})$ is at least Δ_3.*

Then, from the definition of the $\ulcorner \varphi \in sl_X(\mathcal{A}) \urcorner$ we get easily the following

Fact 7.4 *If $X \subseteq \mathbf{N}$ is such that $sl_X(\mathcal{N})$ is a complete theory, then $X \notin (\Sigma_1 \cup \Pi_1)$.*

Now, we will construct an example of a complete extension of $sl(\mathcal{N})$ which is indeed Δ_3. Let $\{\varphi_0, \varphi_1, \varphi_2, \ldots\}$ be a recursive enumeration of all sentences in our language with φ_0 being a tautology and let D be a full binary tree labeled in the following way:



We describe a function which on input i returns a sequence $(\varphi_0, \ldots, \varphi_i)$ which is an initial fragment of the leftmost path in D which form a theory consistent with $sl(\mathcal{N})$. A function uses a Σ_2–complete oracle $sl(\mathcal{N})$. Having such a function one can easily see that the complexity of this path is Δ_3.

The function uses a variable P which, at the end, is the output path and some additional variables like γ which is a conjunction of sentences on the path constructed so far. Below, there is a description of the function.

```
input : n
begin
  P := ∅; γ := φ_0; i := 0;
  while (i < n) {
    i := i + 1;
    if ⌜γ ⇒ φ_i⌝ ∈ sl(N) then {
      P := P⌢(φ_i);
      γ := γ ∧ φ_i;
    }
    else {
      P := P⌢(¬φ_i);
      γ := γ ∧ ¬φ_i;
    }
  }
end
output : P
```

Since the oracle is Σ_2 the constructed function is in Δ_3. But now, if we want to check whether φ_i belongs to the leftmost path in D which is consistent with $sl(\mathcal{N})$ we need only to check whether φ_i is on the path outputed on the input i. This proves that the complexity of this path is also in Δ_3.

8 Generalization

Assume now that $\mathcal{A}$ is an arbitrary infinite relational structure having no more than finite number of constants. Let say, $\mathcal{A} = (A, R_0, R_1, \ldots, R_n, a_0, \ldots, a_k)$. For each finite subset $X \subseteq A$ we denote $\mathcal{A}_X = (X, R_0^X, R_1^X, \ldots, R_n^X, a_0, \ldots, a_k)$, where R_j^X denote the restriction of the relation R_j to the set X. So, each infinite structure $\mathcal{A}$ determines a family $FMG(\mathcal{A}) = \{\mathcal{A}_X \ : \ X \subseteq_{fin} |\mathcal{A}| \text{ and } X \text{ contains all constants }\}$.

Let us denote by I the set of all finite subsets of the universe of a structure $\mathcal{A}$ containing all constants of $\mathcal{A}$. Now, we give two definitions of the relation "sentence φ is true in sufficiently large finite structures":

$\varphi \in sl^I(\mathcal{A}) \iff \forall i \in I \exists j \in I \forall k \in I (i \cup j \subseteq k \to \mathcal{A}_k \models \varphi)$

$\varphi \in sl^{II}(\mathcal{A}) \iff \exists j \in I \forall k \in I (j \subseteq k \to \mathcal{A}_k \models \varphi)$

obviously we have: $sl^{II}(\mathcal{A}) \subseteq sl^I(\mathcal{A})$. Now, we will observe that the converse inclusion also holds. We deduce it from our next observation concerning ultraproduct construction.

By a *Frechet–like filter* on I we mean the least filter Fl such that for every $i \in I$, $J_i \in Fl$, where $J_i = \{j \in I \ : \ i \subseteq j\}$.

Fact 8.1 $sl^I(\mathcal{A}) \subseteq \bigcap_{Fl \subseteq U} Th(\Pi_{i \in I}\mathcal{A}_i/U) \subseteq sl^{II}(\mathcal{A})$.

Proof. Let $\varphi \in sl^I(\mathcal{A})$ and $i \in I$. There exists $j \in I$ such that for arbitrary $k \in I$ if $i \cup j \subseteq k$ then $\mathcal{A}_k \models \varphi$. this means that $J_{i \cup j} \subseteq \{s \ : \ \mathcal{A}_s \models \varphi\}$. So, from the Łoś theorem, $\Pi_{i \in I}\mathcal{A}_i/U \models \varphi$ if $Fl \subseteq U$.

Assume now that $\varphi \notin sl^{II}(\mathcal{A})$. Thus for all $j \in I$ there exists $k \in I$ such that $j \subseteq k$ and $\mathcal{A}_k \models \neg\varphi$. Let $J = \{i \in I \ : \ \mathcal{A}_i \models \neg\varphi\}$. We claim that there exists an ultrafilter F such that $Fl \subseteq F$ and $J \in F$. If no, then for some $i_1, i_2, \ldots, i_n \in I$ $J_{i_1} \cap \ldots \cap J_{i_n} \cap J = \emptyset$. But $J_{i_1} \cap \ldots \cap J_{i_n} = J_s$ for some $s \in I$. So, $J_s \cap J = \emptyset$. This means that for each $k \in I$ such that $s \subseteq k$, $k \notin J$, what means that $\mathcal{A}_k \models \varphi$. This is in contradiction with the fact that $\varphi \notin sl^{II}(\mathcal{A})$. This show that there exists an ultrafilter $U \supseteq Fl$ such that $J \in U$. By the Łoś theorem $\Pi_{i \in I}\mathcal{A}_i/U \models \neg\varphi$. So, $\varphi \notin \bigcap_{Fl \subseteq U} Th(\Pi_{i \in I}\mathcal{A}_i/U)$. **Q.E.D**

Corollary 8.2 $sl^I(\mathcal{A}) = sl^{II}(\mathcal{A})$.

So, the above two operations sl^I and sl^{II} coincide. In the following we will denote this operation by sl^G. Obviously, sl and sl^G are not the same operations. They have different domains and, moreover, if $sl(\mathcal{A})$ and $sl^G(\mathcal{A})$ is defined then $sl(\mathcal{A})$ is a set of sentences of a wider signature than the signature of sentences of $sl^G(\mathcal{A})$.

Examples

1. If $\mathcal{A}$ is a structure of the empty signature and $|\mathcal{A}| = \mathbf{N}$ then $sl^G(\mathcal{A}) = sl^-(\mathcal{A})$. So, in that case $sl^G(\mathcal{A}) = Th(\mathcal{A})$.

2. Let $\mathcal{A} = (\mathbf{N}, <)$. We may observe that in that case $sl^G(\mathcal{A}) = Th((\omega + \omega^*, <))$. So, again, as in the first example, $sl^-(\mathcal{A}) = sl^G(\mathcal{A})$. However here $sl^G(\mathcal{A}) \neq Th(\mathcal{A})$.

3. Let $\mathcal{A} = (\mathbf{N}, R)$ where R is a relation ordering the set of natural numbers in the type $\omega + \omega^*$. For example, let R is defined as follows: $nRm \iff$

$$(2 \mid n \wedge 2 \mid m \wedge n \leq m) \vee (2 \not| n \wedge 2 \not| m \wedge m \leq n) \vee (2 \mid n \wedge 2 \not| m).$$

We may observe that again we have: $sl^G(\mathcal{A}) = Th((\omega + \omega^*, <)) = sl^-(\mathcal{A})$ So, in that case we have: $sl^G(\mathcal{A}) = Th(\mathcal{A})$.

4. Let $\mathcal{A} = (\mathbf{N}, R)$, where R is a relation ordering the set of natural numbers in the type $\omega^*{+}\omega$, or in the type η (i.e. the type of the ordering of the rational numbers). As in the last example we have $sl^G(\mathcal{A}) = Th((\omega+\omega^*, <))$. However now $sl^G(\mathcal{A}) \neq Th(\mathcal{A})$.

From the last two examples we have the following

Fact 8.3 *There are structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \not\equiv \mathcal{B}$ and $sl^G(\mathcal{A}) = sl^G(\mathcal{B})$.*

The third example shows that the equality $sl^G(\mathcal{A}) = Th(\mathcal{A})$ can hold not only in the case of empty signature. In general, a relation between $sl^G(\mathcal{A})$ and $Th(\mathcal{A})$ is stronger than between $sl(\mathcal{A})$ and $Th(\mathcal{A})$.

Fact 8.4 *If φ is a Σ_2–sentence and $\varphi \in Th(\mathcal{A})$ then $\varphi \in sl^G(\mathcal{A})$.*

Proof. Let $\varphi \in Th(\mathcal{A})$ be a Σ_2–sentence. So, φ is of the form $\exists x_1 \ldots \exists x_k \forall y_1 \ldots \forall y_m \psi$, where ψ is a quantifier free formula. Thus for some $a_1, \ldots a_k$ $(\mathcal{A}, a_1, \ldots, a_k) \models \forall y_1 \ldots \forall y_m \psi$. Hence for arbitrary substructure $(\mathcal{A}_0, a_1, \ldots, a_k) \subseteq (\mathcal{A}, a_1, \ldots, a_k)$ $(\mathcal{A}_0, a_1, \ldots, a_k) \models \forall y_1 \ldots \forall y_m \psi$. This gives that $\varphi \in sl^G(\mathcal{A})$. **Q.E.D**

Observe that the last result is not true in the case of $sl(\mathcal{A})$ as shows the third example from the second section of the paper.

Corollary 8.5 *If $Th(\mathcal{A})$ is Σ_2–axiomatizable then $sl^G(\mathcal{A}) = Th(\mathcal{A})$.*

Question. Does the converse implication hold?

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